This activity was inspired by a video presented on the PBS Infinite Series Youtube channel. All of the videos on this channel are quite interesting. I would encourage you to check it out. Here is a link: https://youtu.be/uWwUpEY4c8o.

Like Heracles of ancient Greek myth, we are about to slay they hydra. We won't be using a club and torch like Heracles and Iolaus, though. We will be using our mathematical knowledge. Our hydra regenerates by slightly different rules than the serpent of ancient legend. Here are the rules that we will be following for our hydra game. For reference, "above" means further from the body of the hydra and below means closer to the body of the hydra.

1. Heads can be stacked on top of each other, forming branches of any arrangement.
2. We can only chop off a head if it has nothing above it.
3. The hydra is slain when all heads are cut off from the body.
4. Once a head is chopped off, look at the neck connecting the head immediately below where we just cut and the head below that—2 copies of that neck and anything above it regenerate (except the head we cut off).

This all sounds complicated, so let's look at some examples. Let's say that we have a hydra like the one below.
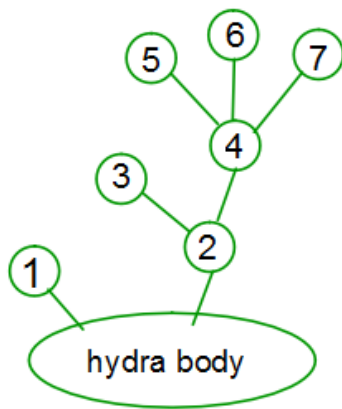
An obvious place to start would be to cut off head 1 since that off would have no consequence—no heads would regenerate. We then have a choice of cutting off heads 3, 5, 6, or 7.

If we cut off head 3, we will copy the neck that connects the body to head 2 and anything above it.

Suppose that instead of head 3, we cut off head 7. The neck between heads 2 and 4, and anything above that, would be copied twice and regenerated. This same result would show up if we cut off either head 5 or head 6 instead.
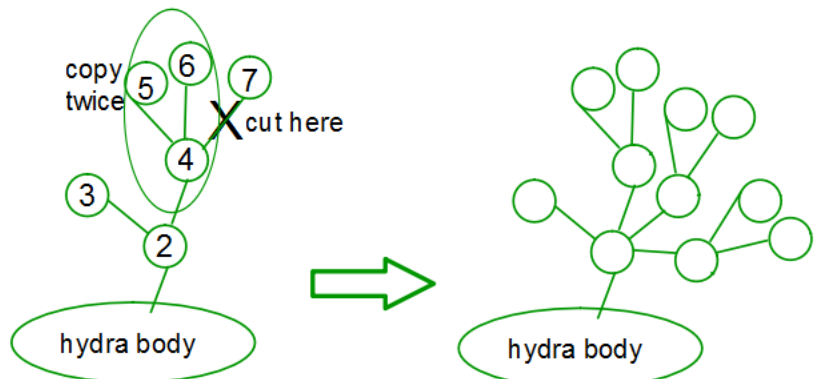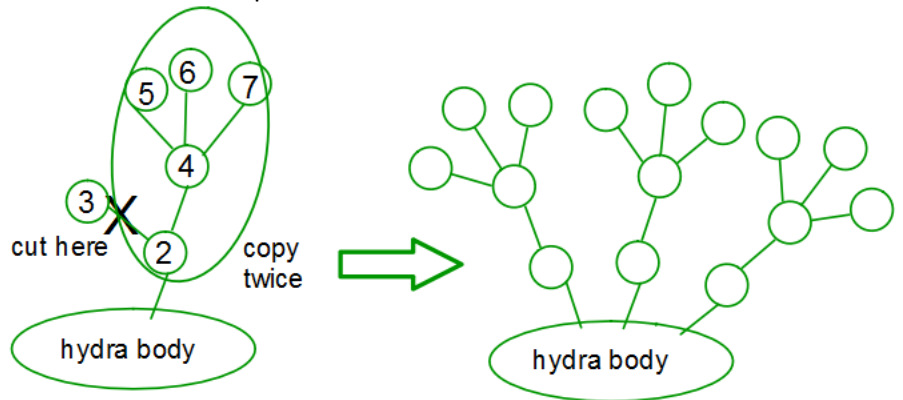
In terms of number of heads, it seems that cutting off head 7 is the better decision. Either way, the number of heads has increased. I would encourage you to use your whiteboards now to finish off this hydra—if you can. Keep track of how many heads you have to cut off in order to accomplish this. See if someone at your table was able to slay the hydra with fewer cuts.

How many heads did you have to cut off? _____

How many heads did your neighbor cut off? _____

Now let's try a different arrangement of hydra. Consider the hydra to the right. Its design is quite a bit simpler than our previous example. There's only one way to start it out since there is only one head that we are allowed to cut off first. Use your white board to sly this hydra, keeping track of how many heads you've cut off. Compare your number with a neighbor and see if anyone was able to do this with fewer cuts.
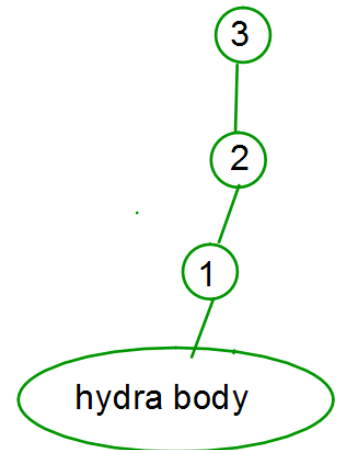
How many heads did you cut off? _____

How many heads did your neighbor cut off? _____

It is entirely possible that you and your neighbor slew the hydra in the same number of cuts for the previous examples. This next one kicks it up a notch. You should have lots of options for how to approach this hydra. See if you can slay it. If you can, keep track of the number of heads you've cut off, then compare that number to your neighbor's.

How many heads did you cut off? _____

How many heads did your neighbor cut off? _____
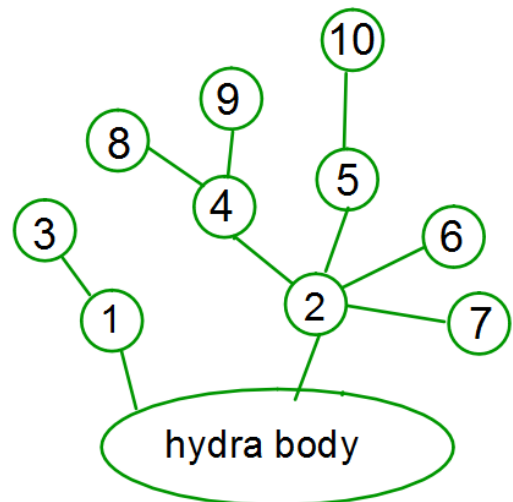
At this stage, you may be thinking to yourself, is there a hydra that cannot be slain? Maybe you even have an idea for one already. In the space below, draw a hydra that you think might be immortal. Once you've drawn it, choose a head to cut off and then draw the result from removing that head. If there is time, you could even challenge a neighbor to slay your hydra.

_____   _____
   Your Hydra   One Head Removed

To explore the notion of whether our hydra can be slain or not, let's see if we can devise a way to quantify the complexity of our hydras. In order to accomplish this, we will use ordinals of infinite numbers. Ordinals are well ordered—that is to say that given any two distinct ordinal numbers, we can say that one of the numbers is defined as smaller than the other. An ordinal number describes the set of all natural numbers (here including 0) that are less than that number. For example, the number 42 describes the set {0, 1, 2, …, 41}. The smallest infinite ordinal is ω. We can operate on ordinal numbers with addition, multiplication, and exponentiation—these operations, however, are not commutative. Since the infinite ordinals are well ordered, the following progression can be written (let n be any natural number).

$$n < \omega < \omega + n < n\omega < \omega^n < \omega^\omega < \omega^{\omega+n} < \omega^{n\omega} < \omega^{\omega^n} < \omega^{\omega^\omega} \ldots$$

Now that we have an understanding of this structure, let's look at how we can use this to describe the complexity of our hydras. Start with the outer-most heads and assign each of them a value of zero. For all heads below these, each head is assigned a value of ω to the power of the head immediately above it. If there are multiple heads sprouting from one, the base head gets an ω to a power for each neck extending upwards and these ω are added together. The complexity of the hydra is the complexity we arrive at for the base of the monster. A hydra with a complexity of 0 has no heads and is slain.

(3) 0
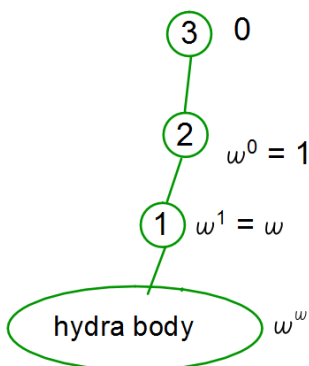
This all sounds complicated again, but let's look at a few examples to help us understand.

(2) $\omega^0 = 1$

First, let's look at our 1-2-3 hydra. The head 3 has a value of 0. Head 2 has a value of ω^0, so actually, just a value of 1. Head 1 has a value of ω^1, which is just ω. Finally, the body has a value of ω^ω and this is the stated complexity of our hydra.

(1) $\omega^1 = \omega$

hydra body ω^ω

Once we cut off head 3, we get a different hydra with some branches. Once again, we assign a value of 0 to each of the outer heads. The lowest head gets a value that is the sum of ω^0 + ω^0 + ω^0, or just plain 3. The body then has a complexity value of ω^3. How does this compare to the original complexity value—from before we cut off a head?

$$w^0 + w^0 + w^0 = 3$$

hydra body w^3
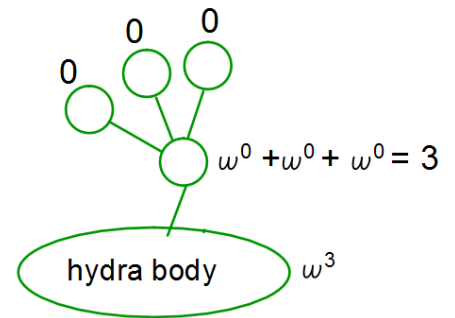
Do you recall the first hydra we looked at and the results when we cut off different heads? Let's take a look at that again and now assign a complexity value to the original and each of the results.

hydra body   hydra body   hydra body
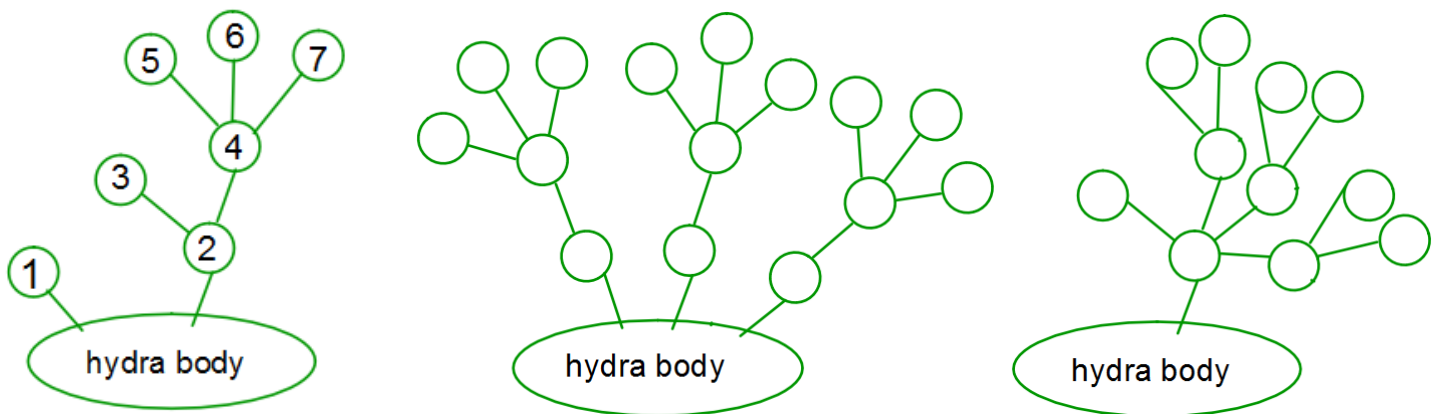
The middle picture comes from removing head 1 then head 3. The right hand picture comes from removing head 1 then head 7 instead. How does the complexity of the original compare to the complexity to each of the others after we have removed a couple of heads? Why was it counterproductive to remove head 3?

Have a look now at the hydra you drew and the result after removing one head. Find the complexity for each of these. How does the complexity of the original compare to that of the one-head-removed version? Did the complexity decrease? If not, is there a head that you could remove so that the complexity would decrease?

The fact is, any hydra we draw can be slain by a sufficient number of cuts. That's not to say that there aren't hydras that will take a whole lot of cuts—more than would be practical for this activity—but any hydra with a finite number of heads can be slain with a finite number of cuts. Not every removable head will reduce the complexity, but there is always a removable head that will reduce the complexity when it is removed. As long as there are heads to remove, we can continue to reduce the hydra's complexity. Eventually the complexity can be reduced to 0 and the hydra is slain. The question I will leave you with, though, is why is there always a head that can be removed that will reduce the complexity of the hydra?